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Hexachords and Their Trichordal Generators: An Introduction

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In this article, bits of analytically and compositionally useful information from numerous sources[1] are brought together with the goal of providing a clear, concise picture of some fundamental aspects of the

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1. The following sources contain lists related to those found here, as well as other relevant information: Allen Forte, "A Theory of Set Complexes for Music," Journal of Music Theory 8/2 (1964): 136-83, The Structure of Atonal Music (New Haven: Yale University, 1973); Howard Hanson, Harmonic Materials of Modern Music (New York: Appleton-Century-Crofts, 1960); Josef Hauer, Vom Melos zur Pauke (Vienna: Universal Edition, 1925) (for a discussion of Hauer's tropes see: Richmond Browne, Review of The Structure of Atonal Music by Allen Forte, JMT 18/2 (1974): 390-409); Donald Martino, "The Source Set and its Aggregate Formations," JMT 5/2 (1961): 224-73, addendum in JMT 6/2 (1962): 322-23; George Perle, Serial Composition and Atonality (Berkeley: University of California, 1962); John Rahn, Basic Atonal Theory (New York: Longman, 1980).

The following sources also contain information relevant to the present discussion: Milton Babbitt, "Some Aspects of Twelve-Tone Composition," The Score and I.M.A. Magazine 12 (1955): 53-61, "Twelve-Tone Invariants as Compositional Determinants," Musical Quarterly 46 (1960): 246-59, reprinted in Problems of Modern Music, ed. Paul Henry Lang (New York: Norton, 1962), "Set Structure as a Compositional Determinant," JMT 5/2 (1961): 72-94, reprinted in Perspectives on Contemporary Music Theory, ed. Benjamin Boretz and Edward Cone (New York: Norton, 1972), "Since Schoenberg," Perspectives of New Music 12/1-2 (1973-74): 3-28; Hubert S. Howe, Jr., "Some Combinational Properties of Pitch Structures," PNM 4/1 (1965): 45-61; Andrew Mead, "Pedagogically Speaking: A Practical Method for Dealing with Unordered Pitch-Class Collections," In Theory Only 7/5-6 (1984): 54-66; Robert D. Morris, "Set Groups, Complementation, and Mappings among Pitch-Class Sets," JMT 26 (1982): 101-144; Robert Morris and Daniel Starr, "A General Theory of Combinatoriality and the Aggregate (Part I)," PNM 16/1 (1977): 3-35, "A General Theory of Combinatoriality and the Aggregate (Part II)," PNM 16/2 (1978): 50-84.

Readers familiar with the Martino article will recognize that portions of it have been used as a starting point and/or model for the present paper. There are a few problems concerning the information that Martino presents about the hexachords and their trichordal generators. Since his article was written prior to a general acceptance of Forte's aggregate/hexachord/trichord interrelationship.[2] The organization is as follows: (1) tables and graphs that catalog hexachord/trichord relationships are presented and explained, (2) some suggestions are offered for use of this information, and (3) a short discussion of the identity/complementation properties of the hexachords is included. Some readers will no doubt find the suggestions and the examples illustrating them familiar territory, but may view the various tables and graphs as handy references. In order to keep the exposition free of sidetracking explanations, a section of definitions of some basic terms and concepts is included at the end.

Tables 1 and 2: Format

Each hexachord can be divided into 10 pairs of trichords.[3] A pair of the same class, [0,1,2] + [0,1,2] for example, is called a single generator; a pair of dissimilar class, such as [0,1,2] + [0,1,5], is called a dual generator.[4] Table 1 lists each of the 50 unique hexachords and their respective trichordal generators.[5] The notation

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labels for set-types, or collection classes, some of Martino's labels are unique to his discussion and therefore less familiar. In particular, the omission of Z-pair complements is confusing, especially in the cataloging of trichordal generators.

2. I would like to acknowledge Andrew Mead for his guidance in theoretical matters for some time prior to and during the writing of this article.

3. For example, with [0,1,2,3,4,5]:

$\{0,1,2\}+\{3,4,5\}$	$\{0,2,3\}+\{1,4,5\}$	$\{0,3,4\}+\{1,2,5\}$	$\{0,4,5\}+\{1,2,3\}$
$\{0,1,3\}+\{2,4,5\}$	$\{0,2,4\}+\{1,3,5\}$	$\{0,3,5\}+\{1,2,4\}$	
$\{0,1,4\}+\{2,3,5\}$	$\{0,2,5\}+\{1,3,4\}$		
$\{0,1,5\}+\{2,3,4\}$			
4 +	3 +	2 +	1 = 10

4. Single generator and dual generator are terms from Martino, "Source Sets." The term "collection class" is equivalent to Forte's "set type."

5. The format for Table 1 is derived from Martino's hexachord table ("Source Sets," p. 229). The prime forms of the hexachords and trichords, as well as the interval vectors of the hexachords, are taken from Forte, "A Theory of Set Complexes for Music." In the tables and graphs of the present paper, the brackets and commas normally used in the notation of collection classes are omitted, as is the prefix number that designates the size or number of elements in a collection class. Brackets, commas, and prefix number are used in labeling hexachords in the body of the paper. In the interest of immediacy Forte's collection class numbers for the trichords are not used.

used for the trichords is that of Martino: 12 = [0,1,2]; the 0, the commas, and the brackets are omitted to conserve space and to facilitate scanning the lists. The order in which the generators are listed is organized as follows: for single generators the criterion for "firstness" is the smaller generator; for dual generators the criteria for "firstness" are: (1) the smaller first half and then (2) the smaller second half. The letters in square brackets to the left of the hexachord label the all-combinatorial hexachords.[6] The angle brackets that normally enclose interval vectors, i.e., <5,4,3,2,1,0>, and the commas that separate ics have been omitted. The superscripts indicate that a generator can be extracted more than once from a hexachord. For example, 6-7 lists the single generator 15^2, which means that this generator--[0,1,5] + [0,1,5]--can be found two distinct ways (i.e., not pc equivalent) within any classic transformation of 6-7.[7] For example, {0,1,2,6,7,8} = {1,2,6} + {7,8,0} and {8,0,1} + {2,6,7}.

In Table 2 the information in Table 1 is presented from the point of view of the trichordal generators; the 78 possible[8] generators are listed[9] along with the hexachords that each will produce. The superscripts indicate that for any single manifestation of a hexachord, the generator in question can be extracted more than once (in other words, the superscripts provide the same information in both Tables 1 and 2). Z-pairs that occur in the list for any one generator are underlined.

Graph X

Graph X provides yet another format for the information in Tables 1 and 2: a graph with the hexachords as the horizontal axis, and the trichordal generators as the vertical axis. The point of reference for Table 1 was the hexachord, and that for Table 2 the trichordal generator; Graph X combines these perspectives in a 'non-listing' representation.

All the properties of Tables 1 and 2 and Graph X are preserved under the cycle of fourths transform.[10] Table 3 lists the M5 or cycle of

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6. The designations A, B, C, etc., are from Babbitt, "Some Aspects."

7. The classic transformations of pitch, called twelve-tone operators, or TTOs, are: transposition, order reversal or inversion, pitch inversion, and combinations of these. See Rahn, Basic Atonal Theory, or Mead, "A Practical Method."

8. There are 12 trichords, thus there are 12 + 11 + 10 . . . + 1 = 78 trichordal pairings, or generators.

9. In Table 2 the order of listing of the generators uses the criteria for the listing of the dual generators of Table 1.

10. The cycle of fourths transform (M5) and the cycle of fifths

fourths transforms for the hexachords and trichords. Hexachords that are M5 mappings of one another have the same number of single and dual generators, and these generators map into one another under M5. For example, 6-1 and 6-32 are M5 mappings of one another. Each has four single generators; those of 6-1: 12, 13, 14, 25 map respectively into 27, 25, 37, 24, of 6-32. The three dual generators of 6-1: $12\text{-}15^2$, $13\text{-}14^2$, $13\text{-}25^2$ map respectively into $15\text{-}27^2$, $25\text{-}37^2$, $13\text{-}25^2$ of 6-32. Two generators that map into one another under M5 have the same number of hexachords, and these hexachords map into one another under M5. For example, the generator 12-12 maps into 27-27. Each has four hexachords listed; those of 12-12: 6-1, Z4, Z6, 7 map respectively into 6-32, Z26, Z38, 7 of 27-27.

This concludes the introduction to the hexachord/trichord interrelationship and its catalogs. In the following sections some aspects of this interrelationship will be examined in more detail with a view toward practical application. Suggested applications are for the most part general enough in nature to allow personal adaptations.

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Some Possibilities

Hexachord Sequences

Comparisons can be made between hexachords to determine shared generators, which may then be used to maintain structural continuity at the trichord level while effecting change at the hexachord level. An example would be the maintenance of generator identity using a sequence of hexachords having one or more common generators. Shared generators can also be used to establish "modulations" between hexachords.[11] Example 1 shows a sequence of such "modulations" in which the initial hexachord (6-2) returns twice via different groups of common generators. These two types of hexachord sequences--maintenance of generator identity, and "modulation" via shared generators--can be used individually or in combination in structuring pitch content.

Hexachordal Hierarchies

A hexachordal hierarchy can be established by using the generators

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transform (M7 yields the inverse of M5) are discussed in Rahn, Basic Atonal Theory, pp. 53-55. These are multiplicative operations, mod. 12. For example, the pc "4" under M5 maps as follows: 4 x 5 = 20; 20 mod. 12 (20 - 12) = 8. The pc mappings under M5 are: 0-0, 1-5, 2-t, 3-3, 4-8, 5-1, 6-6, 7-e, 8-4, 9-9, t-2, e-7.

11. See Babbitt, "Since Schoenberg," for a discussion of this idea as it relates to the all-combinatorial hexachords.

Example 1:

	Shared Generators
6-2 / 6-Z3	12-13, 12-16, 14-15
6-Z3 / 6-5	12-16, 13-15, 13-16
6-5 / 6-Z12	13-16, 15-37, 25-27
6-Z12 / 6-2	12-13, 14-15
6-2 / 6-Z36	12-16, 13-25, 13-36, 14-15
6-Z36 / 6-5	12-14, 12-16, 14-16
6-5 / 6-Z41	15-37, 16-36
6-Z41 / 6-2	12-26, 13-26, 14-15

of a single hexachord as reference; the more generators that a hexachord has in common with the referential hexachord, the higher its status in the hierarchy, or the "closer" to the referential hexachord it is considered to be. For example, 6-20 has four generators: 14^3, 15^3, 37^3, 48. Four hexachords--6-Z4, Z26, Z49, 30--have two generators in common with 6-20; these four hexachords can be considered to have higher status than the numerous hexachords that share only one generator with the referential hexachord; those hexachords with no common generators will have still lower status.

If a limited number of generators are chosen as reference, another type of hexachordal hierarchy is established. If, for example, the single generators 12 and 15 are chosen, the hexachordal hierarchy could be the following:

Generators: 12 and 15

6-Z4, Z6, 7

12 only	15 only
6-1	6-Z38, 8, 9, 14, 20, Z26

All other hexachords

Variations of these forms of hexachordal hierarchy abound.

Example 2:

Type			6-1				
4	X	X	12-12				
	X	X	12-12				
3+1	X	X	12-12				
	X	Y	12-15				
2+2	X	X	12-12	OR	X	Y	12-15
	Y	Y	13-13		Y	X	15-12
2+1+1	X	X	12-12	OR	X	Y	14-13
	Y	Z	13-25		Y	Z	13-25
1+1+1+1	X	Y	12-15				
	W	Z	13-14				

The Trichordal Mosaic

The term "trichordal mosaic" is used here to indicate a group of four trichords that combine to produce an aggregate.[12] The distribution of the four trichords will be one of five types; from maximum similarity to maximum diversity, these are: 4, 3 + 1, 2 + 2, 2 + 1 + 1, and 1 + 1 + 1 + 1. Example 2 shows the trichordal distribution types and illustrates how they may be derived from 6-1; W, X, Y, and Z represent trichord types. Note: a number of hexachords do not have a "single" generator (6-15 and 6-16, for example) and are thus limited in the ways they may be used in these mosaic constructions.

The Hexachord Families of the Trichordal Mosaic

The product of the combination of vertical pairs of trichords or diagonal pairs of trichords in a mosaic will either be the original hexachord(s) (Exx. 3a and 4b), or a pair of "derived" hexachords.[13] Derived hexachords will always be either a dual representation of a single, non-Z-pair hexachord (Exx. 3b and 4a) or a pair of Z-related hexachords (Exx. 3c and 4c) because the derived hexachords are necessarily complementary.

The triangular relationship of the horizontal, vertical, and diagonal hexachords of a mosaic can be represented as in Example 5a.

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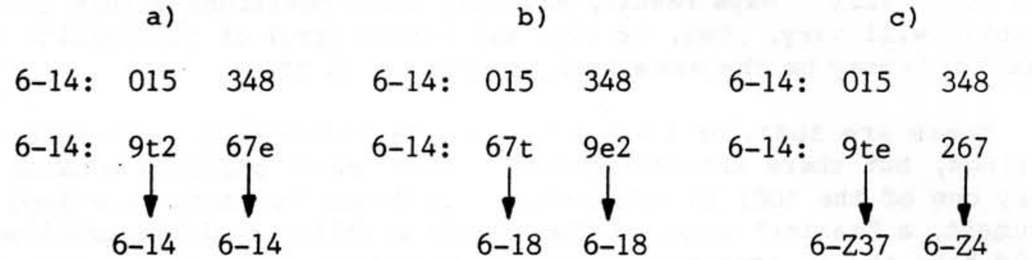
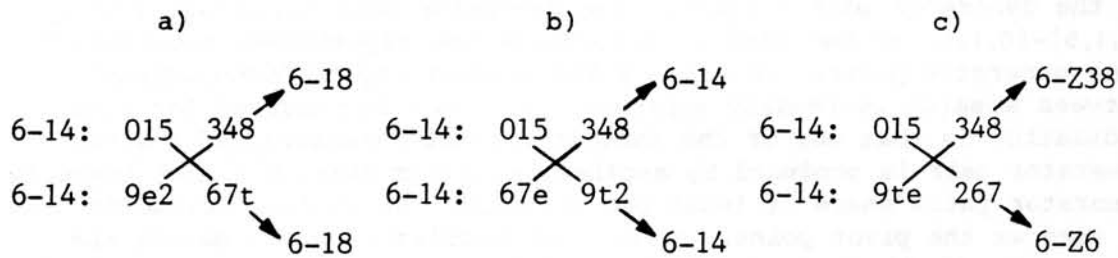
12. Martino uses the term "mosaic" to designate some equal-element division of the 12 pcs, mostly discussing the "trichordal mosaic" and the "tetrachordal mosaic." More generally, the term designates any distribution of the 12 pcs. Non-aggregate producing combinations of collections are discussed by Robert Morris in "Combinatoriality Without the Aggregate," Perspectives of New Music 21/1-2 (1982-83): 432-86.

13. The term "derived hexachords" is adapted from the term "derived harmonic set" in Martino, "Source Set."

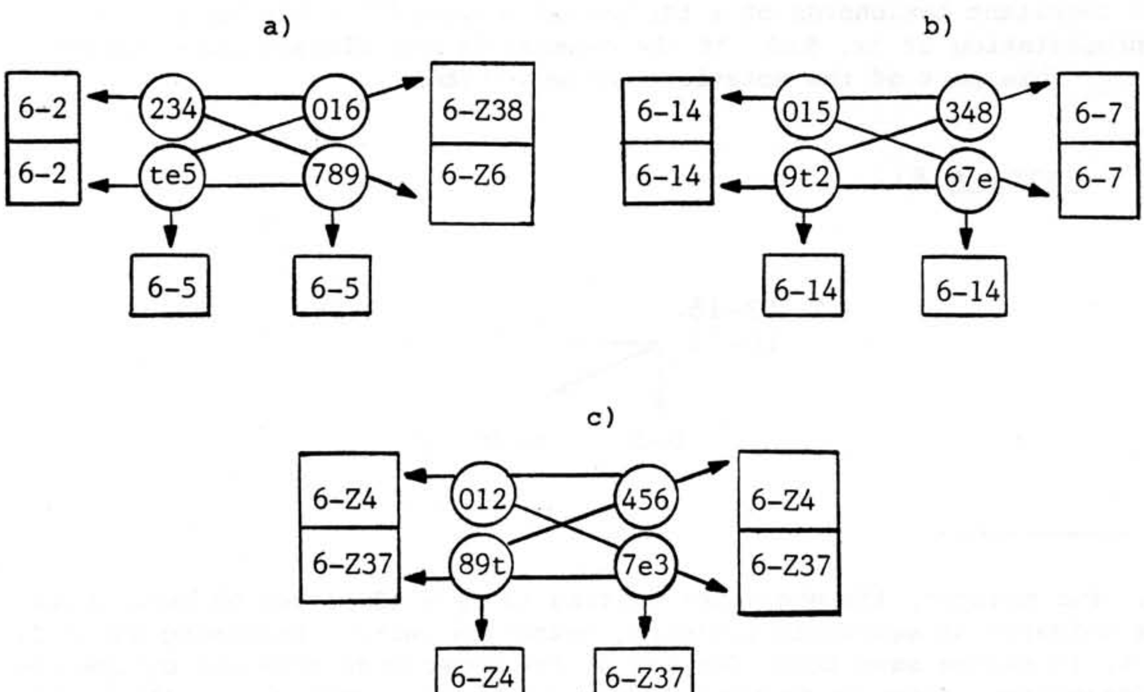
Example 3: Derived Vertical Hexachords



Example 4: Derived Diagonal Hexachords



Example 5:

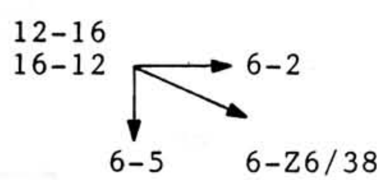


The component trichords can be swapped in any manner and the same three hexachords will always result, although their positions within the triangle will vary. Two, or even all three (rare) of the resultant hexachords may be the same (Exx. 5b and 5c, p. 25).

There are 3081, or 1 + 2 + 3 . . . + 78 trichordal generator pairings, but there are considerably fewer "real" pairings because not every one of the 3081 is available.[14] Although it would be a long document, a "master" graph of the mosaic families could be produced. It would have the 35 aggregate producing hexachord pairs as one axis and the possible generator pairs as the other, with individual generator pairs repeated as necessary to accommodate all of the three-member hexachord families for any given generator. Graph Y is the beginning of this "master" mosaic graph. The three vertical dots indicate omissions in the generator pair listing. The generator pair [0,1,5]-[0,1,5] + [0,1,5]-[0,1,5] is included to illustrate the repetitions required for some generator pairs. In Graph Y the possibility of "modulations" between mosaics is readily apparent. All that is required for such a modulation is that one of the three resultant hexachords of a given generator pair is produced by another generator pair, and that these two generator pairs share at least one trichord. Up to four trichords can be used as the pivot point. Three such modulatory pivot points are circled in Graph Y. (It would be useful in such a graph to be able to indicate from which hexachord pairs the generators as listed can be derived.)

Example 6 shows a shorthand notation suggested by Andrew Mead for the resultant hexachords of a trichordal mosaic.[15] (This is a representation of Ex. 5a.) If the generators are already known to the users, this part of the notation may be omitted.

<u>Example</u> <u>6</u>:



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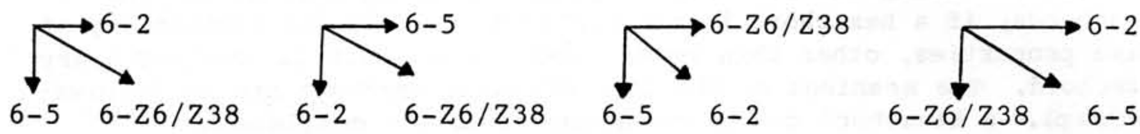
14. For example, the generator pairing 12-12 + 12-13 has no reality in the universe of aggregate producing hexachord pairs. Examining Table 2, p. 1, it can be seen that, for any of the hexachords produced by the 12-12 generator, there is no complementary hexachord produced by the 12-13 generator. Thus 12-12 + 12-13 is not a "real" generator pairing.

15. In conversation with this author.

Example 7:



Mosaic Uses

In reviewing Example 5a, a musical projection of the three hexachords of the mosaic might be the following: horizontal hexachords--registral separation, vertical hexachords--temporal proximity, and diagonal hexachords--orchestration. As suggested earlier, rotation of the four trichords maintains the resultant hexachords, but might suggest different musical projections. Some of the results of trichordal rotation as applied to Example 5a are shown in Example 7 using shorthand notation.

Another strategy for pitch structure might be the maintenance of one of three hexachords of a mosaic while varying the others. In Example 8, the horizontal hexachord (6-2 from Ex. 5a) continues from aggregate to aggregate while the vertical and diagonal hexachords change; different generators are chosen for the horizontal statements of 6-2 in each of the aggregates. Each aggregate might represent any number of the same mosaic, with changes of pitch class being effected by TTOs. For example, a sequence of transpositions may be applied to the first mosaic before progressing to the next mosaic, and so on.

Example 8:

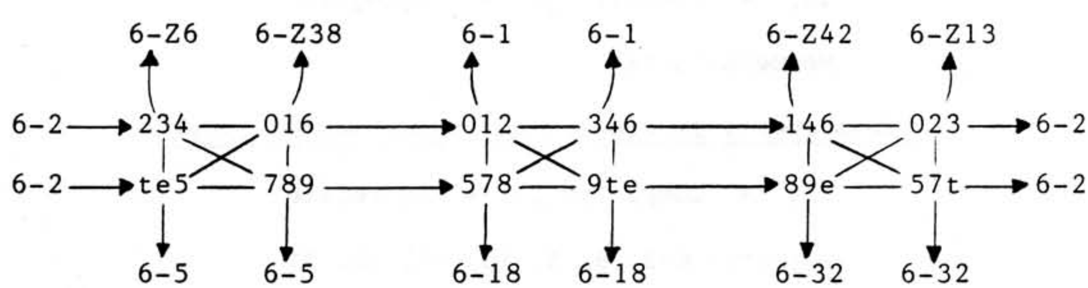


Table 4

Table 4 lists the identity/complementation properties for the hexachords; if a hexachord is not included, it does not exhibit any of these properties, other than Tn=T0, where n=0, which is true for every hexachord. The meanings of the four columnar headings are as follows: Tn=Compl.--a hexachord can be transposed into its complement, TnI=Compl.--a hexachord can be inverted and then transposed into the complement of the original form of the hexachord, TnI=T0--a hexachord can be inverted and then transposed into the original form of the hexachord. The entries in the columns of the properties represent 'n' in each case. For example, in the Tn=Compl. column, 6-1 lists the number 6; if a projection of 6-1, {0,1,2,3,4,5}, is transposed by 6 half-steps the result is the complement {6,7,8,9,t,e}.

If aggregate producing hexachordal combinatoriality[16] is viewed as, very simply, the combination of some transpositional/inversional form of a hexachord with its complement, then given the columnar classifications of Table 4 the combinatorial relationship of maximum pc redundancy may be defined as follows:

$$(Tn_1 = Tn_2I) + Compl.(Tn_3 = Tn_4I) = Aggregate,$$

where n_{1-4} are transposition variables.

The hexachords that satisfy this statement are the all-combinatorial hexachords; they all have at least one entry for each column of Table 4. (For Tn=T0, n=0 for every hexachord.) Two sub-properties of this maximum pc redundancy relationship are the cases where a hexachord combines with (1) a transposition of itself, or (2) with some transposition of an inversion of itself to produce an aggregate. Stated as before, these are:

Sub-property I (Prime Combinatoriality)

$$Tn_1 + Compl.(Tn_2) = Aggregate$$

Members: 6-14

Sub-property II (Inversional Combinatoriality)

$$Tn_1 + Compl.(Tn_2I) = Aggregate$$

Members: 6-2, 5, 9, 15, 16, 18, 21,

22, 27, 30, 31, 33, 34

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16. Hexachordal combinatoriality is discussed in numerous sources. The reader might consult the articles by Babbitt and Morris and Starr cited in n. 1 above.

For 6-14 (sub-property I), 6-30 of sub-property II, as well as the all-combinatorial hexachords, at least two transpositions of the same form of the hexachord contain the same pcs. This reduces the number of unique aggregate producing hexachord pairings for these hexachords from the maximum twelve.[17] The equivalent pairings are at a distance of some interval that divides the chromatic into equal sized units of consecutive pcs.[18] For example, with 6-1 the equivalent pairings are a tritone apart, thus reducing the number of unique aggregate producing pairings by one-half, from 12 to 6. The more pc equivalent forms of a hexachord there are, the fewer unique aggregate producing pairings there will be. For example, 6-20 has three equivalent transpositions of each form that are major thirds apart--dividing the chromatic into three equal units of four consecutive pcs: 12 / 3 = 4 unique hexachords and only two unique aggregate producing hexachord pairings. This redundancy can be used to advantage to create pitch structures like Example 9 (p. 30). In this example the transpositions of the Z-pair are derived from 6-Z6. Note that these transpositions reflect the transpositions of 6-20 that are pc equivalent: T0, T4, T8; T1, T5, T9; T2, T6, Tt; and T3, T7, Te.

Returning for a moment to Table 4, there are 14 hexachords that are not members of the categories discussed so far. They form a group with the following characteristics: (1) each hexachord exhibits TnI = T0--each can be inverted into itself at some transposition; (2) each hexachord is a member of a Z-pair--7 pairs in all. As before, this produces a reduction in the number of unique aggregate producing hexachord pairings--in this case, Z-pairings. For example, for 6-Z4/37 there are two pc equivalent hexachord pairings: 6-Z4,T0: {0,1,2,4,5,6} + 6-Z37,T7: {7,8,9,t,e,3} is pc equivalent to 6-Z4,T6I: {6,5,4,2,1,0} + 6-Z37,TeI: {e,t,9,8,7,3}. This redundancy reduces the total number of unique aggregate producing hexachord pairings by one-half from 24 to 12.

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17. For each hexachord there are 48 forms: 12 prime, 12 retrograde, 12 inversional, and 12 retrograde inversional. For each hexachord T0 = RTn for at least one value of n, and T0I = RTnI for at least one value of n; this equivalence of forms reduces the total of 48 by one-half to 24. For hexachords that are retrograde combinatorial only (Z3/36, Z10/39, Z11/40, Z12/41, Z17/23, Z19/44, Z24/46, Z25/47) 24 is the number of unique aggregate producing hexachord pairs, as there are 12 prime and 12 inversional forms for each half of the Z-pair. For all other hexachords there are at least two aggregate producing hexachord pairs that are pc equivalent, thus further reducing the unique pairings by at least one-half to 12. As described in the text that follows, some hexachords have even fewer unique aggregate producing hexachord pairings.

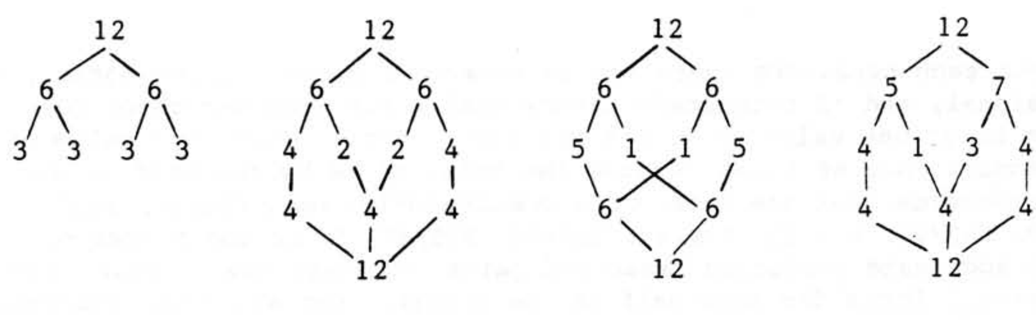
18. See the discussion of "cliques" in Morris and Starr, "General Theory of Combinatoriality."

Example 9:

6-20,T0:	015	489 ➞ 459	801 ➞ 891	045	
6-20,T2:	267	te3 ➞ 6te	237 ➞ t23	67e	
	T0:6-Z6	6-Z38	T4:6-Z6	6-Z38	T8:6-Z6 6-Z38
6-20,T1:	126	59t ➞ 56t	912 ➞ 9t2	156	
6-20,T3:	378	e04 ➞ 7e0	348 ➞ e34	780	
	T1:6-Z6	6-Z38	T5:6-Z6	6-Z38	T9:6-Z6 6-Z38
6-20,T2:	237	6te ➞ 67e	t23 ➞ te3	267	
6-20,T0:	489	015 ➞ 801	459 ➞ 045	891	
	T2:6-Z6	6-Z38	T6:6-Z6	6-Z38	Tt:6-Z6 6-Z38
6-20,T3:	348	7e0 ➞ 780	e34 ➞ e04	378	
6-20,T1:	59t	126 ➞ 912	56t ➞ 156	9t2	
	T3:6-Z6	6-Z38	T7:6-Z6	6-Z38	Te:6-Z6 6-Z38

Example 10:

a) b) c) d)



Conclusion

This entire presentation has focused on only one manifestation of a larger concept: that of set/subset or collection/sub-collection. In the viewpoint taken here the aggregate divides into two hexachords, each of which in turn divides into two trichords (Ex. 10a). But what about other ways of dividing the aggregate? Examples 10b, 10c, and 10d show some other possible divisions of the aggregate with some possible recombinations that could serve to establish collectional relationships. All of these examples are based on use of the total chromatic, but what about using something less than the aggregate as reference?[19] The relationships between and among such collections and sub-collections are as fixed and knowable as those presented here for the aggregates/hexachords/trichords. The integer model of pitch, with its accompanying alphanumeric representation, allows an ease of discovery and a precision in cataloging these relationships--relationships that yield multiple visions of an unchanging chromatic universe.

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Some Definitions

These definitions purposely employ a rather casual language. For more formal definitions, including theorems and proofs, of these and other elements of atonal theory the reader is invited to consult either or both of the following: Rahn, Basic Atonal Theory; Forte, Structure of Atonal Music. For a "practical method for dealing with unordered pc collections" see Mead, "A Practical Method."

pitch class (pc). All enharmonically equivalent pitches (for example, all C#s, Dbs, etc.) are members of the same pitch class. The term was coined by Milton Babbitt.
aggregate. An aggregate is a representation of the total chromatic--all 12 pcs.
alphanumeric pitch notation. The use of numbers and letters to represent pcs. If C is arbitrarily set = 0, then an ascending chromatic scale would read: 0,1,2,3,4,5,6,7,8,9,t,e with t = 10 and e = 11 representing Bb and B respectively. The letters A and B are sometimes used to represent 10 and 11, or Bb and B, if C = 0.
collection. An unordered group of pcs.
collection class. Any group of unordered pcs that are equivalent under transposition and/or inversion are representatives of the same collection class. For example, {C,Db,Eb}, {F,Gb,Ab}, {D,E,F}, {Eb,Db,C}, and {Db,C,Eb} are members of the same collection class. A collection class is represented, using square brackets, as follows: [0,1,3]. (Each of the groups of pitches above belong to the [0,1,3] collection class.) Collection classes are presented in prime form, with pcs packed as closely as possible from left to right, ascending within the octave above one member of the collection which is initialized = 0. For pcs, the size of a collection class may be from 0 to 12 elements, although 0 elements = null set, and 12 elements = aggregate. Collection classes of 2, 3, 4, and 6 elements (dyads, trichords, tetrachords, and hexachords

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19. See Morris, "Combinatoriality Without the Aggregate."

respectively) allow a division of the aggregate into equal-size groups of pcs, and so hold a particular fascination for musicians.

complementation. Within the total chromatic (all 12 pcs), the remaining pcs not included in a collection are the complement of that collection.

interval class (ic). Intervals of the same size (3 semitones or minor third, for example) and their inversions (9 semitones or major sixth = inversion of 3 semitones) are considered to be equivalent and belong to the same interval class. The 6 ics are:

ic 1	1 semitone,	11 semitones
ic 2	2 semitones,	10 semitones
ic 3	3 semitones,	9 semitones
ic 4	4 semitones,	8 semitones
ic 5	5 semitones,	7 semitones
ic 6	6 semitones	

Ics are unordered, denoting no direction between pcs: {C,Eb} = {Eb,C} = ic 3.

interval vector. When all the ics of a collection are tabulated, collated according to ic size, and displayed from left to right in order of ascending ic size, the result is the interval vector. For example, the collection class [0,1,3]--say {C,Db,Eb}--has an interval vector of <1,1,1,0,0,0>. This means there is one ic 1-- {C,Db}, one ic 2--{Db,Eb}, one ic 3-- {C,Eb}, and no cases of ic 4, 5, or 6. This term is Allen Forte's.

Z-related collection classes--sometimes called Z-pairs. This term was coined by Allen Forte to designate a pair of collection classes that have the same interval vector but are not equivalent under transposition and/or inversion. Hexachords that are Z-related are complementary. For example, [0,1,2,3,5,6] and [0,1,2,3,4,7] are Z-related hexachords; extracting {0,1,2,3,5,6} from the total chromatic leaves {4,7,8,9,t,e} which is {0,1,2,3,4,7} inverted and transposed.

all-combinatorial hexachords. The following description is John Rahn's: The "all-combinatorial" hexachords fulfill each of the four criteria for the four kinds of hexachordal combinatoriality. They must: (1) map into themselves under Tn ("retrograde combinatoriality"); (2) map into themselves under TnI ("retrograde inversional combinatoriality"); (3) map into their complements under Tn ("prime combinatoriality"); (4) map into their complements under TnI ("inversional combinatoriality"). The first two criteria mean transpositional and inversional symmetry, (Conditions 3 and 4 entail that the hexachord and its complement cannot be "Z-related.") Every hexachord satisfies (1) under T0. To discover which of the four kinds of combinatoriality some given hexachord possesses, and for which operations, you need only apply the common-tone theorems. (A hexachord maps into itself with 6 pcs in common; it maps into its complement with 0 pcs in common.)

Only six hexachord types satisfy all four criteria. Three hexachord types satisfy all criteria for 'one' value of "n" each; these are called (following Milton Babbitt) "first-order all-combinatorial." The "second-order all-combinatorial hexachord type" satisfies all four criteria for 'two' values of "n" each; the "third-order" for 'three' values of "n" each; and the "sixth-order" (sometimes called the "fourthorder") for 'six' values of "n" each. They are:

first-order	a. [0,1,2,3,4,5]
	b. [0,2,3,4,5,7]
	c. [0,2,4,5,7,9]
second-order	d. [0,1,2,6,7,8]
third-order	e. [0,1,4,5,8,9]
sixth-order	f. [0,2,4,6,8,t]

(Basic Atonal Theory, pp. 117-18). The four criteria are further discussed in connection with Table 4 of the present paper.

Table 1 - p. 1

Hex.	Prime Form	Interval Vector	Single Generators	Dual Generators
[A] 1	012345	543210	12,13,14,24	12-15^2, 13-14^2, 13-25^2
2	012346	443211	14	12-13, 12-16, 12-26, 13-24, 13-25, 13-26, 13-36, 14-15, 24-25
Z3	012356	433221	25	12-13, 12-16, 13-14, 13-15, 13-16, 14-15, 14-36, 15-25, 24-26
Z36	012347	433221		12-14, 12-16, 12-37, 13-25, 13-27, 13-36, 13-37, 14-15, 14-16, 24-26
Z4	012456	432321	12,14,15,25	13-16^2, 14-26^2, 15-24^2
Z37	012348	432321		12-15^2, 12-48, 13-26^2, 13-37^2, 14-16^2, 24-27
5	012367	422232	26	12-14, 12-16, 13-15, 13-16, 14-16, 15-16, 15-37, 16-36, 25-27
Z6	012567	421242	12,15,16,26	14-16^2, 15-27^2, 16-25^2
Z38	012378	421242	15,16,26,27	12-15^2, 13-16^2, 16-37^2
[D] 7	012678	420243	12,15^2,16^2,27	16-26^4
[B] 8	023457	343230	13,15,24,25	12-27^2, 13-37^2, 14-25^2
9	012357	342231	15	12-24, 12-27, 13-16, 13-25, 13-26, 14-37, 16-25, 24-27, 25-26
Z10	013457	333321	13	12-16, 13-27, 14-15, 14-24, 14-26, 14-37, 15-26, 24-37, 25-36
Z39	023458	333321		12-26, 12-37, 13-14, 13-37, 13-48, 14-36, 15-16, 15-25, 24-25, 25-26
Z11	012457	333231		12-13, 13-16, 13-27, 14-25, 14-27, 14-37, 15-16, 15-25, 24-26, 25-36
Z40	012358	333231		12-25, 12-37, 13-15, 13-36, 13-37, 14-37, 15-16, 16-25, 24-26, 25-27
Z12	012467	332232		12-13, 13-16, 14-15, 15-37, 16-24^2, 16-25, 16-26, 25-27, 26-36
Z41	012368	332232	16	12-25, 12-26, 13-26, 13-27, 14-15, 15-37, 16-36, 25-26, 26-27
Z13	013467	324222	13,14,26,36	13-16^2, 14-16^2, 25-37^2
Z42	012369	324222	14,16,25,26	12-36^2, 13-37^2, 15-36^2
14	013458	323430	15	12-15, 13-14, 13-37, 14-25, 14-37^2, 15-27, 24-48, 25-37
15	012458	323421		12-14, 13-15, 13-37, 14-26, 14-36, 14-48, 15-16, 15-26, 24-37, 25-37

Table 1 - p. 2

Hex.	Prime Form	Interval Vector	Single Generators	Dual Generators
16	014568	322431		12–15, 13–14, 14–16, 14–26, 15–24, 15–27, 15–48, 16–37, 25–37, 26–37
Z17	012478	322332		12–14, 13–15, 14–16, 15–25, 16–24, 16–26, 16–37, 16–48, 26–36, 27–37
Z43	012568	322332	16	12–13, 14–15, 14–26, 15–26^{2}, 15–37, 16–36, 25–27, 26–37
18	012578	322242	26	12–13, 14–15, 15–16, 15–25, 16–25, 16–27, 16–36, 16–37, 27–37
Z19	013478	313431	37	13–14, 13–15, 14–15^{2}, 14–16, 15–16, 26–48, 27–37, 36–37
Z44	012569	313431	14	12–14, 14–36, 15–16, 15–25, 15–37^{2}, 16–37, 25–37, 26–48
[E] 20	014589	303630	14^{3},15^{3},37^{3},48	
21	023468	242412		12–26, 13–24, 13–26, 14–26, 14–48, 15–26, 16–26, 24–25, 24–37, 26–36
22	012468	241422		12–24, 13–26, 14–26, 15–24, 15–48, 16–26^{2}, 24–27, 25–26, 26–37
Z23	023568	234222	13,16,25,36	13–26^{2}, 14–37^{2}, 25–26^{2}
Z45	023469	234222		12–36, 13–25^{2}, 14–37^{2}, 16–26^{2}, 24–36^{2}, 27–36
Z24	013468	233331		13–15, 13–24, 13–26, 14–25, 14–27, 15–16, 25–37, 25–48, 26–27, 36–37
Z46	012469	233331	25	12–25, 13–36, 14–24, 14–37, 15–26, 15–37, 16–27, 24–37, 26–37
Z25	013568	233241	13	13–15, 15–25, 15–37, 16–25, 16–27, 24–26, 25–27, 25–37, 36–37
Z47	012479	233241		12–25, 13–25, 14–25, 14–27, 15–37, 16–27, 16–37, 24–26, 25–36, 27–37
Z26	013578	232341	13,15,27,37	15–24^{2}, 16–25^{2}, 26–37^{2}
Z48	012579	232341		12–24, 14–25^{2}, 15–27^{2}, 16–37^{2}, 25–26^{2}, 27–48
27	013469	225222	16,26	13–14, 13–25, 13–36, 14–36, 14–37, 25–36, 25–37, 36–37
Z28	013569	224322		13–14^{2}, 15–36^{2}, 16–26^{2}, 24–36, 25–37^{2}, 36–48
Z49	013479	224322	14,16,36,37	13–25^{2}, 14–26^{2}, 26–37^{2}

Table 1 - p. 3

	Hex.	Prime Form	Interval Vector	Single Generators	Dual Generators
[	Z29	013689	224232	13,16,26,37	$14\text{–}25^2$, $15\text{–}36^2$, $27\text{–}36^2$
[	Z50	014679	224232	25,26,36,37	$13\text{–}14^2$, $16\text{–}25^2$, $16\text{–}37^2$
	30	013679	224223	13,14,25,37	$16\text{–}26^2$, $16\text{–}36^2$, $26\text{–}36^2$
	31	013589	223431		13–14, 14–24, 14–25, 15–16, 15–25, 15–26, 26–37, 27–37, 36–37, 37–48
[C]	32	024579	143250	24,25,27,37	$13\text{–}25^2$, $15\text{–}27^2$, $25\text{–}37^2$
	33	023579	143241	37	13–24, 13–25, 15–37, 16–27, 24–25, 25–26, 25–27, 25–36, 26–27
	34	013579	142422		13–24, 14–24, 15–26, 16–26, 24–25, 25–26, 26–27, 26–36, 26–37, 37–48
[F]	35	02468t	060603	24^3, 26^6, 48	

Table 2 - p. 1

Generator	Hexachord
12-12	1, Z4, Z6, 7
12-13	2, Z3, Z11, Z12, Z43, 18
12-14	Z36, 5, 15, Z17, Z44
12-15	1^2, $Z37^2$, $Z38^2$, 14, 16
12-16	2, Z3, Z36, 5, Z10
12-24	9, 22, Z48
12-25	Z40, Z41, Z46, Z47
12-26	2, Z39, Z41, 21
12-27	8^2, 9
12-36	$Z42^2$, Z45
12-37	Z36, Z39, Z40
12-48	Z37
13-13	1, 8, Z10, Z13, Z23, Z25, Z26, Z29, 30
13-14	1^2, Z3, Z39, 14, 16, Z19, 27, $Z28^2$, $Z50^2$, 31
13-15	Z3, 5, Z40, 15, Z17, Z19, Z24, Z25
13-16	Z3, $Z4^2$, 5, $Z38^2$, 9, Z11, Z12, $Z13^2$
13-24	2, 21, Z24, 33, 34
13-25	1^2, 2, Z36, 9, $Z45^2$, Z47, 27, $Z49^2$, 32^2, 33
13-26	2, $Z37^2$, 9, Z41, 21, 22, $Z23^2$, Z24
13-27	Z36, Z10, Z11, Z41
13-36	2, Z36, Z40, Z46, 27
13-37	Z36, $Z37^2$, 8^2, Z39, Z40, $Z42^2$, 14, 15
13-48	Z39

Generator	Hexachord
14-14	1, 2, Z4, Z13, Z42, Z44, 20^3, Z49, 30
14-15	2, Z3, Z36, Z10, Z12, Z41, Z43, 18, $Z19^2$
14-16	Z36, $Z37^2$, 5, $Z6^2$, $Z13^2$, 16, Z17, Z19
14-24	Z10, Z46, 31, 34
14-25	8^2, Z11, 14, Z24, Z47, $Z48^2$, $Z29^2$, 31
14-26	$Z4^2$, Z10, 15, 16, Z43, 21, 22, $Z49^2$
14-27	Z11, Z24, Z47
14-36	Z3, Z39, 15, Z44, 27
14-37	9, Z10, Z11, Z40, 14^2, $Z23^2$, $Z45^2$, Z46, 27
14-48	15, 21
15-15	Z4, Z6, Z38, 7^2, 8, 9, 14, 20^3, Z26
15-16	5, Z39, Z11, Z40, 15, 18, Z19, Z44, Z24, 31
15-24	$Z4^2$, 16, 22, $Z26^2$
15-25	Z3, Z39, Z11, Z17, 18, Z44, Z25, 31
15-26	Z10, 15, $Z43^2$, 21, Z46, 31, 34
15-27	$Z6^2$, 14, 16, $Z48^2$, 32^2
15-36	$Z42^2$, $Z28^2$, $Z29^2$
15-37	5, Z12, Z41, Z43, $Z44^2$, Z46, Z25, Z47, 33
15-48	16, 22

Table 2 - p. 2

Generator	Hexachord
16–16	Z6, Z38, 7^2, Z41, Z42, Z43, Z23, 27, Z49, Z29
16–24	$Z12^2$, Z17
16–25	$Z6^2$, 9, Z40, Z12, 18, Z25, $Z26^2$, $Z50^2$
16–26	7^4, Z12, Z17, 21, 22^2, $Z45^2$, $Z28^2$, 30^2, 34
16–27	18, Z46, Z25, Z47, 33
16–36	5, Z41, Z43, 18, 30^2
16–37	$Z38^2$, 16, Z17, 18, Z44, Z47, $Z48^2$, $Z50^2$
16–48	Z17
24–24	1, 8, 32, 35^3
24–25	2, Z39, 21, 33, 34
24–26	Z3, Z36, Z11, Z40, Z25, Z47
24–27	Z37, 9, 22
24–36	$Z45^2$, Z28
24–37	Z10, 15, 21, Z46
24–48	14
25–25	Z3, Z4, 8, Z42, Z23, Z46, Z50, 30, 32
25–26	9, Z39, Z41, 22, $Z23^2$, $Z48^2$, 33, 34
25–27	5, Z40, Z12, Z43, Z25, 33
25–36	Z10, Z11, Z47, 27, 33
25–37	$Z13^2$, 14, 15, 16, Z44, Z24, Z25, 27, $Z28^2$, 32^2
25–48	Z24
26–26	5, Z6, Z38, Z13, Z42, 18, 27, Z29, Z50, 35^6
26–27	Z41, Z24, 33, 34
26–36	Z12, Z17, 21, 30^2, 34
26–37	16, Z43, 22, Z46, $Z26^2$, $Z49^2$, 31, 34
26–48	Z19, Z44
27–27	Z38, 7, Z26, 32
27–36	Z45, $Z29^2$
27–37	Z17, 18, Z19, Z47, 31
27–48	Z48
36–36	Z13, Z23, Z49, Z50
36–37	Z19, Z24, Z25, 27, 31
36–48	Z28
37–37	Z19, 20^3, Z26, Z49, Z29, Z50, 30, 32, 33
37–48	31, 34
48–48	20, 35

Table 3: M5 Mappings

	Hex.	Mapping
[A]	1	32
	2	33
[	Z3	Z25
[	Z36	Z47
[	Z4	Z26
[	Z37	Z48
	5	18
[	Z6	Z38
[	Z38	Z6
[D]	7	7
[B]	8	8
	9	9
[	Z10	Z46
[	Z39	Z24
[	Z11	Z40
[	Z40	Z11
[	Z12	Z12
[	Z41	Z41
[	Z13	Z50
[	Z42	Z29
	14	14
	15	31
	16	16
[	Z17	Z17
[	Z43	Z43

	Hex.	Mapping
	18	5
[	Z19	Z44
[	Z44	Z19
[E]	20	20
	21	34
	22	22
[	Z23	Z23
[	Z45	Z45
[	Z24	Z39
[	Z46	Z10
[	Z25	Z3
[	Z47	Z36
[	Z26	Z4
[	Z48	Z37
	27	27
[	Z28	Z28
[	Z49	Z49
[	Z29	Z42
[	Z50	Z13
	30	30
	31	15
[C]	32	1
	33	2
	34	21
[F]	35	35

Trichord	Mapping
[0,1,2]	[0,2,7]
[0,1,3]	[0,2,5]
[0,1,4]	[0,3,7]
[0,1,5]	[0,1,5]
[0,1,6]	[0,1,6]
[0,2,4]	[0,2,4]
[0,2,5]	[0,1,3]
[0,2,6]	[0,2,6]
[0,2,7]	[0,1,2]
[0,3,6]	[0,3,6]
[0,3,7]	[0,1,4]
[0,4,8]	[0,4,8]

Self-Mapping Trichords:

15, 16, 24, 26, 36, 48

Others map as follows:

12-----27

13-----25

14-----37

<u>Graph X</u> - p. 1

Hex.	Trichordal Generators	12-12	12-13	12-14	12-15	12-16	12-24	12-25	12-26	12-27	12-36	12-37	12-48	13-13	13-14	13-15	13-16	13-24	13-25	13-26	13-27	13-36	13-37	13-48	14-14	14-15	14-16	14-24	14-25	14-26	14-27	14-36	14-37	14-48	15-15	15-16	15-24	15-25	15-26	15-27
1	012345	X			2X									X	2X				2X						X															
2	012346		X			X			X									X	X	X		X			X	X														
Z3	012356		X			X									X	X	X									X						X						X		
Z36	012347			X		X						X							X		X	X	X			X	X													
Z4	012456	X															2X								X					2X					X		2X			
Z37	012348				2X								X							2X			2X				2X													
5	012367			X		X										X	X										X									X				
Z6	012567	X																									2X								X					2X
Z38	012378				2X												2X																		X					
7	012678	X																																	2X					
8	023457									2X				X									2X						2X						X					
9	012357						X			X							X		X	X													X		X					
Z10	013457					X								X							X					X		X		X			X						X	
Z39	023458								X			X			X								X	X								X				X		X		
Z11	012457		X														X				X								X		X		X			X		X		
Z40	012358							X				X				X						X	X										X			X				
Z12	012467		X														X									X														
Z41	012368							X	X											X	X					X														
Z13	013467													X			2X								X		2X													
Z42	012369										2X												2X		X															
14	013458				X										X								X						X				2X		X					X
15	012458			X												X							X							X		X		X		X			X	
16	014568				X										X												X			X							X			X
Z17	012478			X												X											X											X		
Z43	012568		X																							X				X									2X	
18	012578		X																							X										X		X		
Z19	013478														X	X										2X	X									X				
Z44	012569			X																					X							X				X		X		
20	014589																								3X										3X					
21	023468								X									X		X										X				X					X	
22	012468						X													X										X							X			
Z23	023568													X						2X													2X							
Z45	023469										X								2X														2X							
Z24	013468															X		X		X									X		X					X				
Z46	012469							X														X						X					X						X	
Z25	013568													X		X																						X		
Z47	012479							X											X										X		X									
Z26	013578													X																					X		2X			
Z48	012579						X																						2X											2X
27	013469														X				X			X										X	X							
Z28	013569														2X																									
Z49	013479																		2X						X					2X										
Z29	013689													X															2X											
Z50	014679														2X																									
30	013679													X											X															
31	013589														X													X	X							X		X	X	
32	024579																		2X																					2X
33	023579																	X	X																					
34	013579																	X										X											X	
35	02468t																																							
		12-12	12-13	12-14	12-15	12-16	12-24	12-25	12-26	12-27	12-36	12-37	12-48	13-13	13-14	13-15	13-16	13-24	13-25	13-26	13-27	13-36	13-37	13-48	14-14	14-15	14-16	14-24	14-25	14-26	14-27	14-36	14-37	14-48	15-15	15-16	15-24	15-25	15-26	15-27

Graph X - p. 2

15-36	15-37	15-48	16-16	16-24	16-25	16-26	16-27	16-36	16-37	16-48	24-24	24-25	24-26	24-27	24-36	24-37	24-48	25-25	25-26	25-27	25-36	25-37	25-48	26-26	26-27	26-36	26-37	26-48	27-27	27-36	27-37	27-48	36-36	36-37	36-48	37-37	37-48	48-48	Hex.	
											X																												1	012345
												X																											2	012346
													X					X																					Z3	012356
													X																										Z36	012347
																		X																					Z4	012456
														X																									Z37	012348
	X							X												X				X															5	012367
			X		2X																			X															Z6	012567
			X						2X															X					X										Z38	012378
			2X			4X																							X										7	012678
											X							X																					8	023457
					X									X					X																				9	012357
																X					X																		Z10	013457
												X							X																				Z39	023458
													X								X																		Z11	012457
					X								X							X																			Z40	012358
	X			2X	X	X														X						X													Z12	012467
	X		X					X											X						X														Z41	012368
																						2X		X									X						Z13	013467
2X			X															X						X															Z42	012369
																	X					X																	14	013458
																X						X																	15	012458
		X							X													X					X												16	014568
				X		X			X	X																X					X								Z17	012478
	X		X					X												X							X												Z43	012568
					X		X	X	X															X							X								18	012578
																												X			X			X		X			Z19	013478
	2X								X													X						X											Z44	012569
																																				3X		X		20	014589
						X						X				X										X													21	023468
		X				2X								X					X								X												22	012468
			X															X		2X														X						Z23	023568
						2X									2X															X									Z45	023469
																						X		X			X									X					Z24	013468
	X						X									X		X									X												Z46	012469
	X				X		X						X							X		X												X					Z25	013568
	X						X		X				X								X										X								Z47	012469
					2X																						2X		X							X			Z26	013578
									2X										2X													X							Z48	012579
			X																		X	X		X										X					27	013469
2X						2X									X							2X													X				Z28	013569
			X																								2X						X			X			Z49	013479
2X			X																					X						2X						X			Z29	013689
					2X				2X									X						X									X			X			Z50	014679
						2X		2X										X								2X										X			30	013679
																											X				X			X			X		31	013589
											X							X				2X							X							X			32	024579
	X						X					X							X	X	X				X											X			33	023579
						X						X							X						X	X	X										X		34	013579
											3X													6X														X	35	02468t

15-36	15-37	15-48	16-16	16-24	16-25	16-26	16-27	16-36	16-37	16-48	24-24	24-25	24-26	24-27	24-36	24-37	24-48	25-25	25-26	25-27	25-36	25-37	25-48	26-26	26-27	26-36	26-37	26-48	27-27	27-36	27-37	27-48	36-36	36-37	36-48	37-37	37-48	48-48

Graph Y

Hexachordal Families of the Trichordal Mosaic--Beginning Only

Generator Pairs / Hex.	1	2	Z3/36	Z4/37	5	Z6/38	7	8	9	Z10/39	Z11/40	Z12/41	Z13/42	14	15	16	Z17/43	18	Z19/44	20	21	22	Z23/45	Z24/46	Z25/47	Z26/48	27	Z28/49	Z29/50	30	31	32	33	34	35
12-12 + 12-12	2x						x																												
12-12 + 12-15	x			x		x																													
12-12 + 12-48				3x																															
12-13 + 12-13	x	x																x																	
•																																			
•																																			
•																																			
15-15 + 15-15						x			x											x															
15-15 + 15-15							x							2x																					
15-15 + 15-15							x	x												x															
•																																			
•																																			
•																																			

Table 4: Identity/Complementation Properties

	Hex.	Tn=Compl.	TnI=Compl.	Tn=T0 *	TnI=T0
[A]	1	6	11		5
	2		11		
	Z4				6
	Z37				4
	5		11		
	Z6				7
	Z38				3
[D]	7	3,9	5,11	6	2,8
[B]	8	6	1		7
	9		11		
	Z13				7
	Z42				3
	14	6			
	15		11		
	16		3		
	18		11		
[E]	20	2,6,10	3,7,11	4,8	1,5,9
	21		1		
	22		11		
	Z23				8
	Z45				6
	Z26				8
	Z48				2
	27		11		
	Z28				6
	Z49				4
	Z29				9
	Z50				1
	30		5,11	6	
	31		7		
[C]	32	6	3		9
	33		1		
	34		11		
[F]	35	1,3,5, 7,9,11	1,3,5, 7,9,11	2,4, 6,8,10	0,2,4, 6,8,10

* For every hexachord, "n" always = 0.

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